A SURVEY OF SCIENTIFIC PAPERS BY G.V. KAMENKOV

(OBZOR NAUCHNYKH RABOT G.V. KAMENKOVA)

PMM Vol. 31, No. 1, 1967, pp. 5-14

Georgii Vladimirovich Kamenkov concerned himself with difficult and fundamental problems of aerodynamics, theory of stability of motion, and the theory of nonlinear vibrations.

1. Aerodynamics. Kamenkov's first paper on aerodynamics [1] dealt with the stability of von Kármán vortex streets. His earliest papers in this field are related to the studies of von Kármán and Zhukovskii. In his fundamental paper on the action of a plane-parallel fluid layer on a cylindrical body, T. von Kármán , originator of the vortex theory of drag, proposed that drag be computed on the basis of the momentum imparted by the body to vortex filaments. A basic problem in this theory was that of the stability of vortex street centers. Von Kármán derived the following relationship between the width b of the vortex street and the distance ℓ between the vortices :

$$\cosh(\pi b/\ell) = \sqrt{2}$$

which was supposed to guarantee the stability of the vortex street.

But Zhukovskii, proceeding from other assumptions about the displacement of the vortex centers, obtained a different formula

$$\cosh(\pi b/\ell) = \sqrt{3}$$

It was this contradiction which G. V. Kamenkov dealt with in his paper. He showed that the disagreement of the von Kármán and Zhukovskii stability conditions was due to the fact that they applied to different physical problems. He discovered that the stability conditions obtained by von Kármán and Zhukovskii were incorrect in both cases, since they were based on equations of perturbed motion taken in only the first approximation.

In his paper Kamenkov showed that more careful consideration of the problem with due allowance for the effect of higher-order terms indicates that vortex streets are not stable in the infinitesimal for any relationship between b and ℓ.

Kamenkov's paper [2] contains an original investigation of the unsteady motions of an airplane wing. The problem was first solved by Chaplygin for an infinite wingspan with the assumption of a constant circulation integral over the wing contour. Moreover, in constructing the flow past the wing he assumed the vibrational periods to be small enough to enable him to neglect variations in the circulation integral due to wing vibrations.

In this paper Kamenkov established structural formulas for the resultant and for the moment of air pressure acting on a wing in unsteady motion with variable circulation. Replacing the profile by a system of vortices situated along its midline, Kamenkov obtained a Fredholm equation of the first kind for determining the circulation. For the case where the midline of the profile can be represented by an analytic function $y(x)$, he solved the resulting integral equation in trigonometric series form.

The fundamental studies carried out by Kamenkov in the field of wing theory in the transcritical domain are presented in [7]. As we know, the circulation theory of the wing, originated by N. E. Zhukovskii, enables one to determine a wing's lift with sufficient accuracy only for very small angles of attack not exceeding the critical angle.

The picture of fluid motion in the transcritical domain is quite distinct from that in the subcritical range : the flow past the wing ceases to be irrotational: a region filled with vortices is formed behind the wing, as a result of which the motion becomes unsteady.

Kamenkov solved the problem by assuming that the motion behind the body corresponded to the flow pattern described by von Kármán and obtained an equation for the drag valid for all values of b and l (the meaning of these quantities is the same as in [1]), which are wholly unrelated to the aforementioned relations of von Kármán. In order to determine the required relationship between b and l, Kamenkov made use of the least-energy-change criterion which in the problem about the uniform motion of a solid through a fluid can be replaced by the criterion of extremal drag Q_k. This relationship between b and l, as well as between b and the circulation integral over the contour encompassing one vortex street can be obtained by solving Equations

$$\frac{\partial Q_k}{\partial b} = \frac{\partial Q_k}{\partial l} = 0$$

Kamenkov's subsequent reasoning is based on the assumption that the circulation lost by the wing over the time T is equal to the circulation of a single vortex. The excellent agreement of his theoretical results with the large body of experimental material cited in the paper confirmed the validity of this assumption concerning the circulation loss.

In conclusion, Kamenkov presented general formulas for determining the coefficients of the aerodynamic forces as functions of the Strouhal number and extended his theory to a wing of finite span.

2. Stability of motion. Kamenkov's papers on the stability of motion can be placed under three headings : stability in critical cases, stability of motion over finite time periods, and stability of motion in near-critical cases.

1°. The problem of the stability of motion was posed with utmost generality by Liapunov in his doctoral thesis "The General Problem of the Stability of Motion"(1892). With maximum rigor Liapunov indicated the cases in which the first approximation does, in fact, resolve the question of stability and isolated the so-called special or critical cases of the stability problem which require consideration of the linear terms in the right sides of the differential equations of perturbed motion.

Assigning particular significance to the investigation of special cases of the problem of stability of motion, Liapunov noted that "in each of them the problem assumes a distinctive character, so that there can be no thought of any general methods for its solution which would apply to all such cases".

In his thesis Liapunov considered the simplest of the special cases : that of a single zero root and a pair of purely imaginary roots for the steady motion, and the case of one root equal to unity and two conjugate imaginaty roots $e^{\pm i\omega\lambda}$ equal to unity in absolute value for systems with periodic coefficients with the period ω, assuming that $\lambda\omega/\pi$ is an incommensurable number. Somewhat later Liapunov considered the case of two zero roots with one group of solutions.

Kamenkov's earliest papers on the stability of motion were a development of the theory of stability of motion in special cases.

In [3] he solved the problem of the stability of the integrals of a system of differential equations of the form

$$x^{\cdot} = X(x, y), \qquad y^{\cdot} = Y(x, y) \qquad (1)$$

Here

$$X(x, y) = X^{(m)}(x, y) + X^{(m+1)}(x, y) + \ldots,$$
$$Y(x,y) = Y^{(m)}(x, y) + Y^{(m+1)}(x, y) + \ldots$$

are analytic functions of x and y which vanish for $x = y = 0$.

He proved the following theorems.

Theorem 1. If the differential equations of perturbed motion of form (1) satisfy the conditions that

1) the function $F(x, y) = xY^{(m)} - yX^{(m)}$ is alternating in the Liapunov sense :
2) the function $X^{(m)}/x$ can be made positive provided that $F(x, y) = 0$, then the unperturbed motion is unstable.

Theorem 2. If the differential equations (1) are such that

1) $F(x, y) = xY^{(m)} - yX^{(m)}$ is an alternating function ;
2) $X^{(m)}/x < 0$, $Y^{(m)}/y < 0$ for $F(x, y) = 0$;
3) $X^{(m)} = 0$, $Y^{(m)} = 0$ do not have common branches passing through the origin ;
4) at least one of the Equations

$$y\frac{\partial X^{(m)}}{\partial x} - x\frac{\partial Y^{(m)}}{\partial x} = \sigma Y^{(m)}, \qquad y\frac{\partial X^{(m)}}{\partial y} - x\frac{\partial Y^{(m)}}{\partial y} = \sigma X^{(m)}$$

is not valid if $F(x, y) = 0$ for all $\sigma < 1$, then the unperturbed motion is asymptotically stable.

Theorem 3. If $F(x, y) = xY^{(m)} - yX^{(m)}$ is an alternating function and if

$$I = F(\cos\theta, \sin\theta)\int_0^{2\pi} \frac{X^{(m)}\cos\theta + Y^{(m)}\sin\theta}{F(\cos\theta, \sin\theta)}\, d\theta > 0 \qquad (2)$$

then the motion is unstable. If $I < 0$, it is asymptotically stable.

If $I = 0$, then the forms $X^{(m)}$ and $Y^{(m)}$ do not solve the problem of stability.

In the forms $X^{(m)}$, $Y^{(m)}$ and F appearing in (2) the variables x and y have been replaced by $\cos\theta$ and $\sin\theta$, respectively.

For alternating F and $I = 0$ the problem is solved by means of norms of higher order than $X^{(m)}$ and $Y^{(m)}$.

In (4) Kamenkov solved the problem of the stability of the integrals of the system

$$x^{\cdot} = y, \qquad y^{\cdot} = Y(x, y, x_1, \ldots, x_n), \qquad x_s^{\cdot} = \sum_{k=1}^{n} p_{sk}x_k + X_s(x, y, x_1, \ldots, x_n)$$
$$(s = 1, \ldots, n)$$

under the familiar assumptions about the right sides. This problem was solved by Liapunov in the case of no adjoint system in his paper "Investigation of one of the special cases of the motion stability problem" (1893, Matem. Sb., No. 2, pp. 253-333). The same problem in the presence of an adjoint system was likewise solved by Liapunov, but came to light and was published only in 1963. Not knowing about Liapunov's solution, Kamenkov used another approach to investigate the problem of stability in the case of two zero roots with one group of solutions with an adjoint system. In this paper Kamenkov made wide use of Chetaev's well-known theorem to prove his own theorems, which greatly simplified handling of the problem.

After separating the critical and noncritical variables Kamenkov constructed the Liapunov and Chetaev functions for the complete system as sums of two functions,

$$V(x, y, x_1, \ldots, x_n) = V_1(x, y) + V_2(x_1, \ldots, x_n)$$

Here $V_1(x, y)$ is the Liapunov or Chetaev function for the critical system, and $V_2(x_1, \ldots, x_n)$ is determined by Equation

$$\sum_{i=1}^{n} (p_{i1}x_1 + \ldots + p_{in}x_n) \frac{\partial V_2}{\partial x_i} = \pm (x_1^2 + \ldots + x_n^2)$$

The plus or minus sign is taken in accordance with the sign of V_1'.

Thus, in [4] Kamenkov proved the possibility of reducing the problem of the stability of an $(n+2)$-th order system to that of a second-order system in nonessentially singular cases.

Paper [5] concerns stability of motion in the special case of two zero roots with two groups of solutions in the presence of an adjoint system.

Kamenkov indicated which transformations might be used to reduce the investigation of the stability of an $(n+2)$-th order system to the equivalent problem for a second order system. This transition from considering the stability of the complete system to the investigation of the critical system alone later came to be called the "reduction principle".

After passing to the investigation of the second-order system, Kamenkov proceeded to generalize the results of [3]. He proved the general theorem about stability with respect to the forms $X^{(m)}$ and $Y^{(m)}$ in the case where $F(x, y) = xY^{(m)} - yX^{(m)}$ is an alternating form.

Theorem. If

1) $F(x, y) = xY^{(m)} - yX^{(m)}$ is a function of either alternating or constant sign;
2) $X^{(m)} / x < 0$, $Y^{(m)} / y < 0$ for $F(x, y) = 0$, then the unperturbed motion is asymptotically stable.

In his later papers [10 and 12], Kamenkov replaced condition (2) by the equivalent condition $R_0(x, y) = xX^{(m)} + yY^{(m)} < 0$.

If the expression $R_0(x, y) = 0$ on at least one of the real straight lines $F(x, y) = 0$, then the forms $X^{(m)}$ and $Y^{(m)}$ do not resolve the question of stability, and one must turn to a higher-order form for its resolution. The stability problem in this case becomes exceedingly complex. It was only in a much later paper [10] that Kamenkov succeeded in solving it.

The results of papers [3 to 5] were already being applied in the Thirties in studies of the stability of the lateral and forward motion of aircraft.

In monograph [6] Kamenkov generalized the results of papers [4 and 5] and first investigated stability in the critical cases of one zero and a pair of purely imaginary roots and two pairs of purely imaginary roots under the assumption that the purely imaginary roots $\pm i\lambda_1$, $\pm i\lambda_2$ are such that the sum $(m_1\lambda_1 + m_2\lambda_2) \neq 0$ for any integer m_s satisfying the condition $(m_1 + m_2) \leq N$, where N is the order of forms in the transformed critical system which resolve the question of stability. The restriction $(m_1 + m_2) \leq N$ does not exclude so-called internal resonance in terms of higher order than N.

Kamenkov showed that the problem of stability in the case of one zero and a pair of purely imaginary roots, as well as that of two purely imaginaty roots can, under this condition, be reduced to the problem of stability for two zero roots with two groups of solutions.

He then considered the general case where the determining equation of the system of differential equations of perturbed motion has an m-tuple zero root associated with m groups of solutions: $2p$ purely imaginary roots satisfying the condition

$$m_1\lambda_1 + \ldots + m_p\lambda_p \neq 0 \quad (m_1 + \ldots + m_p \leqslant N)$$

and q roots with negative real parts. Here m_s are integers, including zero.

The next step was to show that this problem can be reduced in nonessentially singular cases to the investigation of the stability of a system with an $(m+p)$-tuple zero root with $m+p$ groups of associated solutions.

He then proved a very important theorem on instability with respect to m-th order forms for systems with an n-tuple zero root with n associated groups of solutions.

T h e o r e m . If the system of Equations

$$x_s^{\cdot} = X_s^{(m)}(x_1, \ldots, x_n) + X_s^{(m+1)}(x_1, \ldots, x_n) + \ldots \qquad (s=1,\ldots,n)$$

is such that Equations $F_{sk} = x_k X_s^{(m)} - x_s X_k^{(m)} = 0$ for any fixed k and $s = 1, \ldots, k-1, k+1, \ldots, n$ have real solutions different from $x_1 = x_2 = \ldots = x_n = 0$, and if the form

$$R = \sum_{s=1}^{n} x_s X_s^{(m)}(x_1, \ldots, x_n) \qquad \text{for } F_{sk} = 0$$

can assume positive values, then the unperturbed motion is unstable.

To prove this theorem Kamenkov generalized the familiar theorem of Briot and Bouquet for systems of the form

$$x\frac{dy_i}{dx} = Y_i(x, y_1, \ldots, y_n) \qquad (i=1,\ldots,n) \tag{3}$$

where Y_i are holomorphic functions of x, $y_1, \ldots, y_n$ which vanish when $x = y_1 = \ldots = y_n = 0$. He proved that if Equation

$$|p_{sk} - \delta_{sk}\varkappa| = 0 \quad \text{to} \quad p_{sk} = \left.\frac{\partial Y_s}{\partial y_k}\right|_{x=y_1=\ldots=y_n=0}$$

has no positive roots, then there will always be a certain system of holomorphic functions $y_1, \ldots, y_n$ of the variable x which satisfy system (3) and vanish for $x = 0$.

When one is investigating the stability of systems with purely imaginary roots satisfying the condition $\Sigma m_s\lambda_s \neq 0$, Equations $F_{sk} = 0$ always have a nontrivial solution and the theorem on instability is especially important for such systems.

In the fifth and final chapter of his study, Kamenkov investigated those special cases of the problem of stability of motion for systems with periodic coefficients which can be reduced to the cases already considered. He indicated the transformation which can be used to pass from the investigation of periodic stability of motion to the study of stability of equilibrium.

G. V. Kamenkov's last papers on the theory of stability in critical cases [10 and 12] deal with the stability of periodic motions. He considered systems of Equations

$$x_s^{\cdot} = -\lambda_s y_s + X_s(x_1, \ldots, x_p; y_1, \ldots, y_p; \tau), \; y_s^{\cdot} = \lambda_s x_s + Y_s(x_1, \ldots, x_p; y_1, \ldots, y_p; \tau)$$
$$(s = 1, \ldots, p)$$

where X_s and Y_s are holomorphic functions of the variables $x_1, \ldots, x_p$; $y_1, \ldots, y_p$ whose expansion begins with terms of not lower than the second order. The coefficients in the expansions of X_s and Y_s are periodic functions with the common real period ω.

For $p = 1$ and an irrational $\lambda\omega/\pi$ the stability problem had already been solved by Liapunov. In [10] Kamenkov solved the problem for a rational $\lambda\omega/\pi$. He showed that in this case transformations which do not alter the stability problem can be used to reduce it to the investigation of the critical case of two zero roots with two groups of solutions of the form

$$\begin{aligned} x' &= X^{(m)}(x, y) + \ldots + X^{(m+N)}(x, y) + X^{(m+N+1)}(x, y, t) + \ldots \\ y' &= Y^{(m)}(x, y) + \ldots + Y^{(m+N)}(x, y) + Y^{(m+N+1)}(x, y, t) + \ldots \end{aligned} \tag{4}$$

where $m \geq 2$, $m + N = N_1$, N_1 being an arbitrarily large number.

As already noted, the problem of the stability of system (4) in the case where consideration of the forms $X^{(m)}$ and $Y^{(m)}$, suffices for its solution was investigated thoroughly by Kamenkov in his papers [3, 5 and 6]. He noted, however, that the forms $X^{(m)}$ and $Y^{(m)}$, like the first approximation, do not always resolve the question of stability, so that consideration of higher-order forms becomes necessary. In this case the stability problem involves exceptional difficulties which Kamenkov succeeded in overcoming in [10], where he formulated several general theorems on stability not only with respect to $(m + 1)$-th order forms, but for higher-order forms as well.

The application of these theorems to canonical systems enabled him to generalize the results of Levi-Civita and K. L. Zigel'.

The second part of [10] deals with the stability of motion in the critical cases of p pairs of purely imaginary and n zero roots. Kamenkov showed that the problem of stability in the critical case of an arbitrary number of imaginary roots can always be reduced to the investigation of stability in the critical case of a multiple zero root. The multiplicity of the zero root and the number of associated solution groups is determined by the character of the purely imaginary roots. If the purely imaginary roots $\pm i\lambda_s$ satisfy the relation $\Sigma m_s \lambda_s \neq 0$ for irrational λ_s, then by virtue of a substitution, equivalent with respect to the stability problem, Kamenkov arrived at a system of p zero roots with p solution groups, having halved the order of the initial system. If only the roots $\pm i\lambda_s$ are such that $\lambda_s = \alpha_s/\beta_s$ (α_s, β_s are integers), then, as was shown by Kamenkov, it is possible to obtain a system with zero roots of the same order $2p$ as the initial system.

In [10] Kamenkov developed a new form of differential equations of perturbed motion in the critical case of n zero roots with n solution groups for which the construction of Liapunov and Chetaev functions is simplified substantially. He also formulated stability and instability criteria for such equations.

Paper [13], which follows this survey in the present issue, likewise concerns the stability of periodic motions. In it Kamenkov proved a general theorem whereby the problem of stability of periodic motion in nonessentially singular cases can always be reduced to a problem of equilibrium stability.

The theorem on the existence of the holomorphic functions $z_j = z_j(y_1, \ldots, y_{n_1}; t)$, periodic in t satisfying the system (see p. 19 of paper [13])

$$\frac{\partial z_j}{\partial t} + \sum_{s=1}^{n_1} \frac{\partial z_j}{\partial y_s}(g_{s1}y_1 + \ldots + g_{sn_1}y_{n_1} + Y_s) = p_{j1}z_1 + \ldots + p_{jp}z_p + Z_j$$

enabled Kamenkov to investigate the stability of motion in certain essentially singular cases as well.

2°. The investigation of stability of motion over a finite time interval reduces first of all to the formulation of the concept of stability which, like Liapunov's definition, would express a natural quality of motion in the sense of strength and nonyielding with respect to the initial perturbations, but over a finite time interval only.

This was the definition proposed by Kamenkov in his paper [8].

If the differential equations of perturbed motion

$$x_s^{\cdot} = p_{s1}(t)x_1 + \ldots + p_{sn}(t)x_n + X_s(x_1, \ldots, x_n; t) \qquad (s=1,\ldots,n) \tag{5}$$

(where p_{si} are real, continuous, and bounded functions of time t, and the expansion of the functions X_s in integer positive powers of x_s begins with terms of not lower than the second order) are such that for a sufficiently small positive number A the quantities x_s considered as functions of time satisfy the condition

$$\sum_{i=1}^{n} (a_{i1}x_1 + \ldots + a_{in}x_n)^2 \leqslant A$$

over the finite time interval $[t_0, t_0 + T]$, provided that the initial values x_{s0} of these functions satisfy the condition

$$\sum_{i=1}^{n} (a_{i1}x_{10} + \ldots + a_{in}x_{n0})^2 \leqslant A, \qquad \det \| a_{\lambda\mu} \| \neq 0 \qquad (\lambda, \mu = 1, \ldots, n)$$

then the unperturbed motion is stable during the time interval T; in the contrary case it is unstable, i.e. $T = 0$.

Operating with this definition of stability, Kamenkov formulated and proved the following fundamental theorems [8].

Theorem 1. If the characteristic equation corresponding to a system of differential equations of perturbed motion at $t = t_0$ has no multiple roots, but only negative roots, or complex roots with negative real parts, then the unperturbed motion is stable over some finite time interval T.

Theorem 2. If the characteristic equation has at least one positive root or two positive roots with positive real parts, the unperturbed motion is not stable over a finite time interval, i.e. $T = 0$.

Theorem 3. If the characteristic equation has at least one zero or two purely imaginary roots, the rest of the roots being either negative or complex with negative real parts, then the unperturbed motion may turn out to be unstable over a finite time interval.

In proving these theorems Kamenkov represented the initial system (5) in the form

$$x_s^{\cdot} = p_{s1}(t_0)x_1 + \ldots + p_{sn}(t_0)x_n + \Delta p_{s1}(t)x_1 + \ldots + \Delta p_{sn}(t)x_n + X_s(x_1, \ldots, x_n; t)$$
$$(s=1,\ldots,n)$$

(where $p_{si}(t) = p_{si}(t_0) + \Delta p_{si}(t)$, $\Delta p_{si}(t_0) = 0$ and $p_{si}(t_0)$ are the values of the functions $p_{si}(t)$ for $t = t_0$) and then transformed the linear part of the system with constant coefficients to canonical form.

The paper also contains a method for determining the time interval T over which the motion is stable.

The formulation of the problem of stability of motion over a finite time interval and the method of its solution given by Kamenkov, turned out to be exceptionally useful in the solution of a number of practical problems. His paper [8] opened a whole new line of research on the problem of stability of motion.

3°. In developing the general theory of stability of motion in critical cases, Kamenkov in his paper [9] came to grips with one of its most obscure aspects — that of the stability of motion in the near-critical cases so important to the engineer, especially in dealing with the problems of controlled flight.

The term "near-critical problems" refers to cases where the characteristic equation

$$D(\varkappa) = \det \| p_{\lambda\mu} - \delta_{\lambda\mu}\varkappa \| = 0 \qquad (6)$$
$$(\lambda, \mu = 1, \ldots, n;\ \delta_{\lambda\mu} = 0 \text{ for } \lambda \neq \mu, \quad \delta_{\lambda\mu} = 1 \text{ for } \lambda = \mu)$$

corresponding to a system of ordinary differential equations of perturbed motion of the form

$$x_s^{\bullet} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s(x_1, \ldots, x_n) \qquad (7)$$

has in addition to its negative real parts at least one root with a small positive or negative real part.

Kamenkov formulated this problem of stability of motion in highly original fashion in his paper [9]. The fact is that the existence of a root of equation (6) with a positive real part is usually a sufficient condition for unstable motion regardless of the terms $X_s(x_1, \ldots, x_n)$. It is also clear that the existence of no other roots save those with negative real parts guarantees the stability of unperturbed motion. These two remarkable theorems of Liapunov which form the basis of the entire theory of linear vibrations of mechanical systems and the stability of their motion were derived under some very rigid limitations as regards the initial perturbations $x_{10}, x_{20}, \ldots, x_{n0}$.

In defining the stability of motion Liapunov assumed that all $|x_{s0}|$ can be made smaller than any specified number. Thus, the limiting value of the initial perturbations is zero. Noting the fact that under real conditions the initial perturbations may well be bounded from below, Kamenkov defined the stability of unperturbed motion in the following way [9].

If the space $x_1, \ldots, x_n$ contains a closed region G with the property that the perturbations $x_1, \ldots, x_n$ considered as functions of time and satisfying equations of perturbed motion (7) do not go beyond this region for any values of $t \geq t_0$ provided that the initial conditions $x_{10}, \ldots, x_{n0}$ lie inside this region or on its boundary, then the unperturbed motion is stable ; otherwise it is unstable.

The above definition of stability given by Kamenkov does not violate the physical significance with which Liapunov invested his definition of the stability of motion . Kamenkov's definition is also close to that given by Poincaré in the third mémoire of his monograph "On Curves Defined by Differential Equations" .

In addition to resolving the fundamental problem of stability, Kamenkov proposed a method for finding the region G — a matter of practical interest.

In his paper Kamenkov attacked the problem of nonlinear vibrations, showing the profound connection between this problem and Liapunov functions. The same line of research was later pursued by Kamenkov in paper [11].

3. Nonlinear vibrations. For the last few years of his life Kamenkov concerned himself with the problems of nonlinear oscillations. Among such problems considered in [11] were the following : finding the conditions of existence of periodic solutions, investigation of the stability of these solutions, determination of the form of vibrations and processes involved in their establishment.

Kamenkov investigated both autonomous systems of the type

$$dx/dt = -\lambda y + \mu X(x, y, z_1, \ldots, z_n, \mu), \quad dy/dt = \lambda x + \mu Y(x, y, z_1, \ldots, z_n, \mu)$$

$$dz_s/dt = \sum_{i=1}^{n} p_{si} z_i + \mu Z_s(x, y, z_1, \ldots, z_n; \mu) \qquad (s = 1, \ldots, n)$$

and nonautonomous systems of the type

$$\frac{dx_s}{dt} = -\lambda_s y_s + \mu X_{s1}(x_1, \ldots, x_n; y_1, \ldots, y_n; t) + \mu^2 X_{s2}(x_1, \ldots, x_n; y_1, \ldots, y_n; t) + \ldots + f_{s0}(t) + \mu f_{s1}(t) + \ldots \qquad (s = 1, \ldots, n)$$

$$\frac{dy_s}{dt} = \lambda_s x_s + \mu Y_{s1}(x_1, \ldots, x_n; y_1, \ldots, y_n; t) + \mu^2 Y_{s2}(x_1, \ldots, x_n; y_1, \ldots, y_n; t) + \ldots + \varphi_{s0}(t) + \mu\varphi_{s1}(t) + \ldots$$

Here μ is a small parameter ; X, Y and Z_s can be expressed as series in the parameter μ whose coefficients are polynomials or arbitrary degree in x, y and z_s : X_{si} , Y_{si} are polynomials of arbitrary degree with continuous coefficients periodic in t with the common period 2π : f_{si} and φ_{si} are continuous periodic functions with the same period 2π.

In [11] Kamenkov formulated and proved the following theorems for the above systems.

Theorem 1 . If the system of initial equations is such that the associated equation

$$L_1(V) = \frac{1}{2\pi}\int_0^{2\pi} [X_1(V\cos\theta, V\sin\theta)\cos\theta + Y_1(V\cos\theta, V\sin\theta)\sin\theta]\, d\theta = 0$$

has k positive roots of odd multiplicity in V, then to each of these roots there corresponds at least one limiting cycle and each of these cycles has a corresponding periodic solution of this system. On the other hand, if the equation $L_1(V) = 0$ has positive real roots of even multiplicity, then the problem of the existence of periodic solutions corresponding to these roots is not solved to first-order terms in μ.

Theorem 2 . If the system of initial equations is such that the equation $L_1(V) = 0$ corresponding to this system has a root $V = V_j$ of odd multiplicity equal to $2k - 1$, and if

$$\frac{d^{2k-1}L_1(V)}{dV^{2k-1}} < 0 \qquad \text{for } V = V_j$$

then the periodic vibrations corresponding to the root $V = V_j$ are stable : if, on the other hand,

$$\frac{d^{2k-1}L_1(V)}{dV^{2k-1}} > 0 \qquad \text{for } V = V_j$$

they are unstable.

Theorem 3 . If the system of initial equations is such that Equation $L_1(V) = 0$ has a root $V = V_j$ of even multiplicity $2k$, then the domain of existence of periodic solutions is unstable . Moreover, if

$$\frac{d^{2k}L_1(V)}{dV^{2k}} < 0 \qquad \text{for } V = V_j$$

the domain is stable with respect to external perturbations and unstable with respect to internal ones : if, on the other hand,

$$\frac{d^{2k}L_1(V)}{dV^{2k}} > 0 \qquad \text{for } V = V_j$$

the domain of existence of periodic solutions is stable with respect to internal perturbations and unstable with respect to external ones.

Kamenkov then solved similar problems for the case where the question is resolved not by first-order terms, but rather by a finite number of terms in μ. He then proceeded to describe a method for determining the μ at which and below which the indicated periodic solutions exist, as well as a method for constructing the periodic solutions in the form of series in powers of the parameter μ. He also investigated the processes of establishment of the resulting periodic solutions (transient processes).

Thus, in his paper [11] Kamenkov presented a general method for investigating vibrations in nonlinear systems based on the use of Liapunov's functions, calling it the "method of Liapunov functions". This method makes it possible to find periodic solutions which are either analytic or nonanalytic with respect to μ even in those cases where the system of initial equations does not become linear for $\mu = 0$.

In the last few years of his life Kamenkov worked on his monograph "Stability and Vibrations of Nonlinear Systems" in which he generalized the results obtained in the aforementioned papers.

V. G. Veretennikov, A. S. Galiullin, S. A. Gorbatenko and A. L. Kunitsyn

PAPERS BY G.V. KAMENKOV

1933

1. O vikhrevoi teorii lobovogo soprotivleniia (On the vortex theory of drag). Tr. III Vses. konferentsii po aerodinamike, 1933 g. (Proceedings of the 3rd All-Union Conference on Aerodynamics, 1933), pp. 171-186. Also in Trudy Kazan. aviats. Inst. No. 2, pp. 33-43, 1934.
2. Neustanovivsheesia dvizhenie kryla aeroplana (Unsteady motion of the airplane wing). Sb. nauchn. Tr. Kazan. aviats. Inst., No. 1, pp. 12-17, 1933.

1935

3. Issledovanie odnogo osobennogo, po Liapunovu, sluchaia zadachi ustoichivosti dvizheniia (Study of a Liapunov special case of problem of stability of motion). Sb. nauchn. Tr. Kazan. aviats. Inst., No. 3, pp. 24-31, 1935.
4. Ob ustoichivosti dvizheniia v odnom osobennom sluchae (On the stability of motion in a special case). Sb. nauchn. Tr. Kazan. aviats. Inst., No. 4, pp. 3-18, 1935.

1936

5. Issledovanie odnogo osobennogo sluchaia zadachi ob ustoichivosti dvizheniia (Study of a special case of the motion stability problem). Sb. nauchn. Tr. Kazan. aviats. Inst., No. 5, pp. 19-28, 1936.

1939

6. Ob ustoichivosti dvizheniia (On the stability of motion). Trudy Kazan. aviats. Inst., No. 9, pp. 3-137, 1939.

1946

7. Teoriia kryla v zakriticheskoi oblasti (Theory of the wing in the transcritical domain). Trudy Kazan. aviats. Inst., No. 18, pp. 3-38, 1946.

1953

8. Ob ustoichivosti dvizheniia na konechnom intervale vremeni (On the stability of motion over a finite time interval). PMM Vol. 17, No. 5, 1953.

1963

9. Ob ustoichivosti dvizheniia v sluchaiakh blizkikh k kriticheskim (On the stability of motion in near-critical cases). Trudy Univ. druzhby narodov im. P. Lumumba, Ser. teor. Mekh., Vol. 1, No. 1, pp. 3-15, 1963.

1965

10. K zadache ob ustoichivosti dvizheniia v kriticheskikh sluchaiakh (On the problem of the stability of motion in critical cases). PMM Vol. 29, No. 6, 1965.

1966

11. Issledovanie nelineinykh kolebanii s pomoshch'iu funktsii Liapunova (Investigation of nonlinear vibrations by means of Liapunov functions). Trudy Univ. druzhby narodov im. P. Lumumba, Ser. teor. Mekh., Vol. 15, No. 3, pp. 3-55, 1966.
12. Issledovanie ustoichivosti periodicheskikh dvizhenii (Investigation of the stability of periodic motions). Trudy Univ. druzhby narodov im. P. Lumumba, Ser. teor. Mekh., Vol. 15, No. 3, pp. 56-81, 1966.

1967

13. Ob ustoichivosti periodicheskikh dvizhenii (On the stability of periodic motions). PMM Vol. 31, No. 1, 1967.

Translated by A. Y.